

# Coherent State Functional Integral in Loop Quantum Cosmology: Alternative Dynamics

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Coherent state functional integral for the minisuperspace model of loop quantum cosmology is studied. By the well-established canonical theory, the transition amplitude in the path integral representation of loop quantum cosmology with alternative dynamics can be formulated through group averaging. The effective action and Hamiltonian with higher-order quantum corrections are thus obtained. It turns out that for a non-symmetric Hamiltonian constraint operator, the *Moyal (star)-product* emerges naturally in the effective Hamiltonian. For the corresponding symmetric Hamiltonian operator, the resulted effective theory implies a possible quantum cosmological effect in large scale limit in the alternative dynamical scenario, which coincides with the result in canonical approach. Moreover, the first-order modified Friedmann equation still contains the particular information of alternative dynamics and hence admits the possible phenomenological distinction between the different proposals of quantum dynamics.

**Keywords:** Loop quantum cosmology; Coherent state functional integral; Effective theory

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## I. INTRODUCTION

In the last two decades, considerable progress has been made in loop quantum gravity (LQG), which is a background independent approach to quantum gravity [1–4]. The starting point of LQG is the Hamiltonian *connection dynamics* of GR rather than the ADM formalism. By taking the holonomy of  $su(2)$ -connection  $A_a^i$  and flux of densitized triad  $E_j^b$  as basic variables, the quantum kinematical framework of LQG has been rigorously constructed, and the Hamiltonian constraint operator can also be well defined to represent quantum dynamics. Moreover, a few physically significant results have been obtained in the minisuperspace models of loop quantum cosmology (LQC) [5, 6]. The most interesting one is the *resolution of big bang singularity* in LQC [7–10]. Besides the canonical formalism, the so-called spin foam models were proposed as the path integral formalism of LQG [1]. However, whether the two approaches are equivalent to each other is a longstanding open question. Thanks to the development of LQC, we have a much simpler theory to address this question. As symmetry-reduced models, there are only finite number of degrees of freedom in LQC. Following the conventional method in quantum mechanics, one can find the path integral formalism of LQC starting with the canonical formalism. This approach has been implemented by a series of works [11–14] with the scheme of simplified LQC [15]. Here one employed the complete basis of eigen-states of the volume operator to formulate a path integral with somehow *descrete-steps*, which inherited certain properties of spin foams [16]. Moreover, the first-order effective action for the path integral was derived by this approach [13, 14], which implied the origin of singularity resolution of LQC in the path integral representation. In canonical LQC, the effective Hamiltonian constraint with higher-order quantum corrections could even be obtained by the semiclassical analysis using coherent states, which implied a possible effect of quantum gravity on large scale cosmology [17–19]. It is thus interesting to see whether the effective Hamiltonian can be confirmed by some path integral representation. Since the higher-order corrections of the Hamiltonian come from the quantum fluctuations, a natural attempt to achieve them is to employ coherent state path integral [20].

In LQC, the Hamiltonian constraint equation is usually presented to Klein-Gordon like equation by coupling with a massless scalar field, where the corresponding gravitational Hamiltonian operator, as some multiplication of several self-adjoint operators, is non-symmetric in the kinematical Hilbert space [9, 10, 15]. While this treatment is essential in order to obtain the physical states satisfying the constraint equation, it also provides elegant physical models to examine the so-called *Moyal \*-product* in quantum mechanics. At the very beginning, Moyal proposed the *\*-product*

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in order to clarify the role of statistical concepts in quantum mechanics system [21]. Then this idea were generalized to many situations including quantum spacetime itself. In canonical quantum theories, the *\*-product* can also be understood by coherent state approach [22]. Thus it is also possible and desirable to derive the *\*-product* in coherent state functional integral approach. This idea has been accomplished in the WDW quantum cosmology and LQC [23]. In these models, the *Euclidean* and *Lorentz* terms in the gravitational Hamiltonian constraint are combined together since they are proportional to each other in spatially flat and homogeneous cases. However, this is impossible in the full theory, where the Lorentz term has to be quantized in a form quite different from the Euclidean one [3, 4]. This kind of quantization procedure which kept the distinction of the two terms was proposed as alternative dynamics for LQC [18]. Hence we will study the coherent state functional integral in spatially flat isotropic FRW cosmology coupled with a massless scalar field  $\phi$  in the alternative quantization framework.

## II. COHERENT FUNCTIONAL INTEGRALS

We consider the following Hilbert-Einstein action of gravity coupled with a massless scalar field:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \phi_{,\nu} \phi_{,\nu}. \quad (2.1)$$

In the spatially flat model of FRW cosmology, we fix a space-like sub-manifold  $S$ , which is topologically  $\mathbb{R}^3$  and equipped with Cartesian coordinates  $x^i (i = 1, 2, 3)$  and a fiducial flat metric  $q_{ab}$ . The physical 3-metric  $q_{ab}$  is then determined by a scale factor  $a$  satisfying  $q_{ab} = a^2 q_{ab}$ . It is convenient to introduce an elementary cell  $\mathcal{V}$  and restrict all integrations to this cell in Hamiltonian analysis. The volume of  $\mathcal{V}$  with respect to  $q_{ab}$  is denoted as  $V_o$ . As in the full loop quantum gravity, we employ the new canonical variables  $(A_a^i, E_i^a)$  [24]. Due to the homogeneity and isotropy, we can fix a set of orthonormal cotriad and triad  $({}^o\omega_a^i, {}^oe_i^a)$  compatible with  ${}^oq_{ab}$  and adapted to  $\mathcal{V}$ . Then the basic canonical variables take the simple form  $A_a^i = cV_o^{-(1/3)} {}^o\omega_a^i$ ,  $E_i^a = p\sqrt{{}^oq} V_o^{-(2/3)} {}^oe_i^a$  and thus are reduced to  $(c, p)$  with the Poisson bracket:  $\{c, p\} = 8\pi G\gamma/3$ , where  $\gamma$  is the Barbero-Immirzi parameter. Following the  $\bar{\mu}$ -*schem* of “improved dynamics” [10], the regulator  $\bar{\mu}$  used in holonomies is given by  $\bar{\mu} = \sqrt{\Delta/|p|}$ , where  $\Delta = 4\sqrt{3}\pi\gamma\ell_p^2$  is a minimum nonzero eigenvalue of the area operator [6]. In order to do the semiclassical analysis, it is convenient to introduce new dimensionless conjugate variables [15, 18]:

$$b := \frac{\bar{\mu}c}{2}, \quad v := \frac{\text{sgn}(p)|p|^{\frac{3}{2}}}{2\pi\gamma\ell_p^2\sqrt{\Delta}}, \quad (2.2)$$

with the Poisson bracket  $\{v, b\} = -\frac{1}{\hbar}$ , where the Planck length  $\ell_p$  is given by  $\ell_p^2 = G\hbar$ . From the matter part of action (2.1), we can get  $p_\phi = \frac{a^3 V_o \dot{\phi}^2}{2}$  and the poisson bracket:  $\{\phi, p_\phi\} = 1$ . The kinematical Hilbert space of the quantum theory is supposed to be a tensor product of the gravitational and matter parts. In LQC, one employed the standard *Schrödinger* representation for matter to construct Hilbert space  $\mathcal{H}_{\text{kin}}^{\text{matt}}$ , while gravity was quantized by the polymer-like representation [24]. Thus quantum states in the gravitational Hilbert space of LQC are functions  $\Psi(v)$  of  $v$  with support on a countable number of points and with finite norm  $\|\Psi\|^2 := \sum_v |\Psi(v)|^2$  [25]. Hence the inner product is defined by a *Kronecker delta*  $\langle v' | v \rangle = \delta_{v', v}$ . The basic operators act on a quantum state  $\Psi(v, \phi)$  in the kinematical Hilbert space  $\mathcal{H}_{\text{kin}}^{\text{grav}}$  as:

$$\hat{v}\Psi(v, \phi) = v\Psi(v, \phi), \quad \widehat{e^{ib}}\Psi(v, \phi) = \Psi(v + 1, \phi). \quad (2.3)$$

To obtain the physical states, one has to solve the quantum Hamiltonian constraint equation:

$$-\hat{C} \cdot \Psi(v, \phi) = \left( -\frac{\hat{p}_\phi^2}{\hbar^2} + \hat{\Theta} \right) \Psi(v, \phi) = 0, \quad (2.4)$$

where  $\hat{\Theta} \equiv \hat{\Theta}_E + \hat{\Theta}_L$  is a positive second-order difference operator defined by:

$$\hat{\Theta}_E \cdot \Psi(v, \phi) = \frac{3\pi G\gamma^2}{4} [v(v+2)\Psi(v+4, \phi) - 2v^2\Psi(v, \phi) + v(v-2)\Psi(v-4, \phi)], \quad (2.5a)$$

$$\hat{\Theta}_L \cdot \Psi(v, \phi) = -\frac{3\pi G(1+\gamma^2)}{16} [v(v+4)\Psi(v+8, \phi) - 2v^2\Psi(v, \phi) + v(v-4)\Psi(v-8, \phi)]. \quad (2.5b)$$

Here we use the *alternative quantization* scheme proposed in [18] in which the *Euclidean* and *Lorentz* terms in the gravitational *Hamiltonian constraint* are treated separately. Together with the simplified treatment in [15], we can get

the operators in Eqs. (2.5a) (2.5b) corresponding to the *Euclidean* and *Lorentz* terms respectively. Solutions to the constraint equations and their physical inner products can be obtained through the group averaging procedure. Now we concern about *coherent state functional integrals*. The (*generalized*) coherent state of the matter part is labeled by a complex variable  $z_o := \frac{1}{\sqrt{2\sigma}}(\phi_o + \frac{i}{\hbar}\sigma^2 p_{\phi_o})$  and defined by

$$|\Psi_{z_o}\rangle := \int_{-\infty}^{\infty} d\phi e^{-\frac{(\phi-\phi_o)^2}{2\sigma^2}} e^{\frac{i}{\hbar}p_{\phi_o}(\phi-\phi_o)} |\phi\rangle, \quad (2.6)$$

which is the eigenstate of the *annihilation* operator  $\hat{z} = \frac{1}{\sqrt{2\sigma}}(\hat{\phi} + \frac{i}{\hbar}\hat{p}_{\phi}\sigma^2)$ , where  $\sigma$  describes the width of the wave-packet or quantum fluctuation. It satisfies the key properties of a coherent state, namely, saturation of Heisenberg's uncertainty relation, resolution of identity and peakness property. On the other hand, due to the *polymer-like* structure, the coherent state of LQC is different from that of the matter part. Here one can define  $\zeta_o = \frac{1}{\sqrt{2d}}(v_o + ib_o d^2)$  to label the *generalized* coherent state [17, 25]:

$$(\Psi_{\zeta_o}| := \sum_{v \in \mathbb{R}} e^{-\frac{(v-v_o)^2}{2d^2}} e^{-ib_o(v-v_o)} (v|, \quad (2.7)$$

where  $d$  is the characteristic *width* of the wave packet and  $1 \ll d \ll v_o$  because of the semiclassical feature. For practical use, one defines the projection of this state on some lattice of variable  $v$ , saying the *shadow state* [25]:

$$|\Psi_{\zeta_o}\rangle^{\text{shad}} := \sum_{k=-\infty}^{\infty} e^{-\frac{(k-v_o)^2}{2d^2}} e^{ib_o(k-v_o)} |k\rangle, \quad k \in \mathbb{Z}, \quad (2.8)$$

where we chose the regular lattice  $\{v = k, k \in \mathbb{Z}\}$ . This shadow state also has the analogous properties of a coherent state. The resolution of identity now reads

$$\int_{-\infty}^{\infty} dv_o \int_{-\pi}^{\pi} \frac{db_o}{2\pi} \frac{|\Psi_{\zeta_o}\rangle \langle \Psi_{\zeta_o}|}{\langle \Psi_{\zeta_o} | \Psi_{\zeta_o} \rangle} = \sum_{k=-\infty}^{\infty} |k\rangle \langle k| \equiv \mathbb{I}, \quad (2.9)$$

where the identity  $\mathbb{I}$  is in the subspace in which the states have support only on the regular lattice. The whole coherent state of LQC reads  $|\Psi_{z_o}\rangle |\Psi_{\zeta_o}\rangle \equiv |\Psi_{z_o}\rangle \otimes |\Psi_{\zeta_o}\rangle$ .

In the path integral of the conventional non-relativistic quantum mechanics, one needs to compute the matrix element of the *evolution operator*  $e^{-i\Delta t \hat{H}}$  within the time interval  $\Delta t$ . However, the situation of cosmology in GR is very different, since we are considering totally constrained systems and the operator  $\hat{C}$  is not a *true Hamiltonian*. Instead, we start from the physical inner product, i.e., the transition amplitude, of coherent states with *normalization*:

$$A([\Psi_f], [\Psi_i]) \equiv \frac{\langle \Psi_{\eta_f} | \langle \Psi_{z_f} | \int_{-\infty}^{\infty} d\alpha e^{i\alpha \hat{C}} | \Psi_{z_i} \rangle | \Psi_{\eta_i} \rangle}{\| \Psi_{\eta_f} \| \| \Psi_{z_f} \| \| \Psi_{z_i} \| \| \Psi_{\eta_i} \|}. \quad (2.10)$$

To calculate the *transition amplitude*, we split a fictitious *time interval*  $\Delta\tau = 1$  into  $N$  pieces  $\epsilon = \frac{1}{N}$ . To deal with the parameter  $\alpha$  in group averaging procedure, we employ the trick in [14] to generalize the one single group averaging to multiple ones, i.e.,

$$\lim_{\alpha_o \rightarrow \infty} \int_{-\alpha_o}^{\alpha_o} d\alpha e^{i\alpha \hat{C}} | \Psi_{\text{kin}} \rangle \quad (2.11)$$

$$= \lim_{\tilde{\alpha}_{N_o}, \dots, \tilde{\alpha}_{1_o} \rightarrow \infty} \frac{1}{2\tilde{\alpha}_{N_o}} \int_{-\tilde{\alpha}_{N_o}}^{\tilde{\alpha}_{N_o}} d\tilde{\alpha}_N \cdots \frac{1}{2\tilde{\alpha}_{2_o}} \int_{-\tilde{\alpha}_{2_o}}^{\tilde{\alpha}_{2_o}} d\tilde{\alpha}_2 \int_{-\tilde{\alpha}_{1_o}}^{\tilde{\alpha}_{1_o}} d\tilde{\alpha}_1 e^{i(\tilde{\alpha}_N + \dots + \tilde{\alpha}_1) \hat{C}} | \Psi_{\text{kin}} \rangle, \quad \forall | \Psi_{\text{kin}} \rangle \in \mathcal{H}_{\text{kin}}. \quad (2.12)$$

In order to trace the power for expansion, we re-scale the parameters by  $\tilde{\alpha}_n = \epsilon \alpha_n (n = 1, \dots, N)$  and thus rewrite the exponential operator as:  $e^{i \sum_{n=1}^N \epsilon \alpha_n \hat{C}} = \prod_{n=1}^N e^{i \epsilon \alpha_n \hat{C}}$ . Inserting  $N$  times of coherent states resolution of identity of  $|\Psi_{z_o}\rangle$  and Eq. (2.9), Eq. (2.10) can be casted into

$$A([\Psi_f], [\Psi_i]) = \lim_{\alpha_{N_o}, \dots, \alpha_{1_o} \rightarrow \infty} \frac{1}{2\alpha_{N_o}} \int_{-\alpha_{N_o}}^{\alpha_{N_o}} d\alpha_N \cdots \frac{1}{2\alpha_{2_o}} \int_{-\alpha_{2_o}}^{\alpha_{2_o}} d\alpha_2 \cdot \epsilon \int_{-\alpha_{1_o}}^{\alpha_{1_o}} d\alpha_1 A_N^{\text{matt}} A_N^{\text{grav}}, \quad (2.13)$$

where

$$A_N^{\text{matt}} = \int_{-\infty}^{\infty} d\phi_{N-1} \dots d\phi_1 \int_{-\infty}^{\infty} \frac{dp_{\phi_{N-1}}}{2\pi\hbar} \dots \frac{dp_{\phi_1}}{2\pi\hbar} \prod_{n=1}^N \frac{\langle \Psi_{z_n} | e^{i\epsilon\alpha_n \frac{\hat{p}_\phi^2}{\hbar^2}} | \Psi_{z_{n-1}} \rangle}{\|\Psi_{z_n}\| \|\Psi_{z_{n-1}}\|}, \quad (2.14a)$$

$$A_N^{\text{grav}} = \int_{-\infty}^{\infty} dv_{N-1} \dots dv_1 \int_{-\pi}^{\pi} \frac{db_{N-1}}{2\pi} \dots \frac{db_1}{2\pi} \prod_{n=1}^N \frac{\langle \Psi_{\eta_n} | e^{-i\epsilon\alpha_n \hat{\Theta}} | \Psi_{\eta_{n-1}} \rangle}{\|\Psi_{\eta_n}\| \|\Psi_{\eta_{n-1}}\|}, \quad (2.14b)$$

with  $z_N \equiv z_f, z_0 \equiv z_i, \eta_N \equiv \eta_f$ , and  $\eta_0 \equiv \eta_i$ . Notice that the characteristic widths  $\sigma$  and  $d$  at different steps are not necessarily the same. So we have to denote  $\sigma_n$  and  $d_n$  in the semiclassical states  $|\Psi_{z_n}\rangle$  and  $|\Psi_{\zeta_n}\rangle$  respectively at the “n-step”. Now the main task is to calculate the matrix elements of the exponential operators on coherent states. The exponential operator  $e^{i\epsilon\alpha_n \hat{C}}$  can be expanded as  $1 + i\epsilon\alpha_n \hat{C} + \mathcal{O}(\epsilon^2)$ . For the purpose of a concise writing, we introduce some intermediate-step notations:

$$\bar{p}_{\phi_n} \equiv \frac{\sigma_n^2 p_{\phi_n} + \sigma_{n-1}^2 p_{\phi_{n-1}}}{\sigma_n^2 + \sigma_{n-1}^2}, \quad \bar{\sigma}_n^2 \equiv \frac{2\sigma_n^2 \sigma_{n-1}^2}{\sigma_n^2 + \sigma_{n-1}^2}.$$

The product of the matrix elements in Eq. (2.14a) can be calculated as [23]:

$$\prod_{n=1}^N \frac{\langle \Psi_{z_n} | e^{i\epsilon\alpha_n \frac{\hat{p}_\phi^2}{\hbar^2}} | \Psi_{z_{n-1}} \rangle}{\|\Psi_{z_n}\| \|\Psi_{z_{n-1}}\|} = \left( \prod_{n=1}^N \frac{\langle \Psi_{z_n} | \Psi_{z_{n-1}} \rangle}{\|\Psi_{z_n}\| \|\Psi_{z_{n-1}}\|} \right) \exp \left[ \frac{i\epsilon\alpha_n}{\hbar^2} \sum_{n=1}^N \left( p_{\phi_{n-1}}^2 + \frac{\hbar^2}{\sigma_n^2 + \sigma_{n-1}^2} \right) \right], \quad (2.15)$$

where the product of series  $\langle \Psi_{z_n} | \Psi_{z_{n-1}} \rangle$  can be expressed as

$$\begin{aligned} \prod_{n=1}^N \frac{\langle \Psi_{z_n} | \Psi_{z_{n-1}} \rangle}{\|\Psi_{z_n}\| \|\Psi_{z_{n-1}}\|} &= \exp \left[ \frac{\phi_N^2 + p_{\phi_N}^2 \sigma_{N+1}^2 \sigma_N^2 / \hbar^2}{2(\sigma_{N+1}^2 + \sigma_N^2)} - \frac{\phi_0^2 + p_{\phi_0}^2 \sigma_1^2 \sigma_0^2 / \hbar^2}{2(\sigma_1^2 + \sigma_0^2)} \right] \left( \prod_{n=1}^N \sqrt{\frac{2\sigma_n \sigma_{n-1}}{\sigma_n^2 + \sigma_{n-1}^2}} \right) \\ &\cdot \exp \left[ \epsilon \sum_{n=1}^N \left( -\frac{2(\sigma_{n+1}^2 + \sigma_n^2) \phi_n \frac{\phi_n - \phi_{n-1}}{\epsilon} - (\sigma_{n+1} + \sigma_{n-1}) \frac{\sigma_{n+1} - \sigma_{n-1}}{\epsilon} \phi_n^2}{2(\sigma_{n+1}^2 + \sigma_n^2)(\sigma_n^2 + \sigma_{n-1}^2)} + \frac{i}{\hbar} \bar{p}_{\phi_n} \frac{\phi_n - \phi_{n-1}}{\epsilon} \right. \right. \\ &\quad \left. \left. - \frac{1}{4\hbar^2} \frac{4(\sigma_{n+1}^2 \sigma_n^2 \sigma_{n-1}^2 + \sigma_n^4 \sigma_{n-1}^2) p_{\phi_n} \frac{p_{\phi_n} - p_{\phi_{n-1}}}{\epsilon} + 2\sigma_n^4 (\sigma_{n+1} + \sigma_{n-1}) \frac{\sigma_{n+1} - \sigma_{n-1}}{\epsilon} p_{\phi_n}^2}{(\sigma_{n+1}^2 + \sigma_n^2)(\sigma_n^2 + \sigma_{n-1}^2)} \right) \right]. \quad (2.16) \end{aligned}$$

Here we introduced a *virtual width*  $\sigma_{N+1}$  by hand, satisfying  $\sigma_{N+1} - \sigma_N = \sigma_N - \sigma_{N-1}$ , in order to get the tidy sum in the exponential position. In the limit of  $N \rightarrow \infty$ ,  $\sigma_{N+1}$  will approach  $\sigma_N \equiv \sigma_f$  and hence does not effect the quantum dynamics.

For the gravitational part, careful calculations outlined in the Appendix yield

$$\begin{aligned} \prod_{n=1}^N \frac{\langle \Psi_{\zeta_n} | e^{-i\epsilon\alpha_n \hat{\Theta}} | \Psi_{\zeta_{n-1}} \rangle}{\|\Psi_{\zeta_n}\| \|\Psi_{\zeta_{n-1}}\|} &= \left( \prod_{n=1}^N \frac{\langle \Psi_{\zeta_n} | \Psi_{\zeta_{n-1}} \rangle}{\|\Psi_{\zeta_n}\| \|\Psi_{\zeta_{n-1}}\|} \right) \exp \left[ i\epsilon\alpha_n \cdot 3\pi G \right. \\ &\times \sum_{n=1}^N \left( \gamma^2 \left( (\bar{v}_n^2 + \frac{\bar{d}_n^2}{2}) (\sin^2(2\bar{b}_n) (1 - \frac{8}{d_n^2 + d_{n+1}^2}) + \frac{4}{d_n^2 + d_{n-1}^2}) + i\bar{v}_n \sin(4\bar{b}_n) \frac{2d_n^2}{d_n^2 + d_{n-1}^2} (1 - \frac{8}{d_n^2 + d_{n+1}^2}) \right) \right. \\ &\quad \left. \left. - \frac{1 + \gamma^2}{4} \left( (\bar{v}_n^2 + \frac{\bar{d}_n^2}{2}) (\sin^2(4\bar{b}_n) (1 - \frac{32}{d_n^2 + d_{n+1}^2}) + \frac{16}{d_n^2 + d_{n-1}^2}) + 2i\bar{v}_n \sin(8\bar{b}_n) \frac{2d_n^2}{d_n^2 + d_{n-1}^2} (1 - \frac{32}{d_n^2 + d_{n+1}^2}) \right) \right) \right] \quad (2.17) \end{aligned}$$

where  $\bar{v}_n \equiv \frac{d_{n-1}^2 v_n + d_n^2 v_{n-1}}{d_n^2 + d_{n-1}^2}$ ,  $\bar{b}_n \equiv \frac{d_n^2 b_n + d_{n-1}^2 b_{n-1}}{d_n^2 + d_{n-1}^2}$ ,  $\bar{d}_n^2 \equiv \frac{2d_n^2 d_{n-1}^2}{d_n^2 + d_{n-1}^2}$ , and

$$\begin{aligned} \prod_{n=1}^N \frac{\langle \Psi_{\zeta_n} | \Psi_{\zeta_{n-1}} \rangle}{\|\Psi_{\zeta_n}\| \|\Psi_{\zeta_{n-1}}\|} &= \exp \left[ \frac{v_N^2 + b_N^2 d_{N+1}^2 d_N^2}{2(d_{N+1}^2 + d_N^2)} - \frac{v_0^2 + b_0^2 d_1^2 d_0^2}{2(d_1^2 + d_0^2)} \right] \left( \prod_{n=1}^N \sqrt{\frac{2d_n d_{n-1}}{d_n^2 + d_{n-1}^2}} \right) \\ &\cdot \exp \left[ \epsilon \sum_{n=1}^N \left( -\frac{2(d_{n+1}^2 + d_n^2) v_n \frac{v_n - v_{n-1}}{\epsilon} - (d_{n+1} + d_{n-1}) \frac{d_{n+1} - d_{n-1}}{\epsilon} v_n^2}{2(d_{n+1}^2 + d_n^2)(d_n^2 + d_{n-1}^2)} + i\bar{b}_n \frac{v_n - v_{n-1}}{\epsilon} \right. \right. \\ &\quad \left. \left. - \frac{4(d_{n+1}^2 d_n^2 d_{n-1}^2 + d_n^4 d_{n-1}^2) b_n \frac{b_n - b_{n-1}}{\epsilon} + 2d_n^4 (d_{n+1} + d_{n-1}) \frac{d_{n+1} - d_{n-1}}{\epsilon} b_n^2}{4(d_{n+1}^2 + d_n^2)(d_n^2 + d_{n-1}^2)} \right) \right]. \quad (2.18) \end{aligned}$$

Now we take the limit  $N \rightarrow \infty$  and substitute  $\int_0^1 d\tau$  for  $\sum_{n=1}^N \epsilon$  to get the functional integral formalism of the amplitude:

$$A([\Psi_f][\Psi_i]) = e^{\frac{1}{2}(|z_f|^2 - |z_i|^2 + |\zeta_f|^2 - |\zeta_i|^2)} \int \mathcal{D}\alpha \int [\mathcal{D}\phi(\tau)][\mathcal{D}p_\phi(\tau)][\mathcal{D}v(\tau)][\mathcal{D}b(\tau)] e^{i(S_\alpha^{\text{matt}} + S_\alpha^{\text{grav}})}, \quad (2.19)$$

where

$$S_\alpha^{\text{matt}} = \int_0^1 d\tau \left( i \frac{d}{d\tau} \left( \frac{\phi^2}{4\sigma^2} \right) + i \frac{d}{d\tau} \left( \frac{\sigma^2 p_\phi^2}{4\hbar^2} \right) + \frac{p_\phi \dot{\phi}}{\hbar} + \frac{\alpha}{\hbar^2} \left( p_\phi^2 + \frac{\hbar^2}{2\sigma^2} \right) \right), \quad (2.20)$$

$$\begin{aligned} S_\alpha^{\text{grav}} = & \int_0^1 d\tau \left( i \frac{d}{d\tau} \left( \frac{v^2}{4d^2} \right) + i \frac{d}{d\tau} \left( \frac{d^2 b^2}{4} \right) + b\dot{v} \right. \\ & + \alpha 3\pi G \left[ \gamma^2 \left( v^2 + \frac{d^2}{2} \right) \left( \left( \sin^2(2b) \left( 1 - \frac{4}{d^2} \right) + \frac{2}{d^2} \right) + iv \sin(4b) \left( 1 - \frac{4}{d^2} \right) \right) \right. \\ & \left. \left. - \frac{1 + \gamma^2}{4} \left( v^2 + \frac{d^2}{2} \right) \left( \left( \sin^2(4b) \left( 1 - \frac{16}{d^2} \right) + \frac{8}{d^2} \right) + 2iv \sin(8b) \left( 1 - \frac{16}{d^2} \right) \right) \right] \right). \end{aligned} \quad (2.21)$$

Here the “dots” over  $\phi$  and  $v$  stand for the *time derivative* with respect to the *fictitious time*  $\tau$ . The *functional measures* are defined on *continuous paths* by taking the limit of  $N \rightarrow \infty$ :

$$\int \mathcal{D}\alpha := \lim_{N \rightarrow \infty} \lim_{\alpha_{N_o}, \dots, \alpha_{1_o} \rightarrow \infty} \frac{1}{2\alpha_{N_o}} \int_{-\alpha_{N_o}}^{\alpha_{N_o}} d\alpha_N \cdots \frac{1}{2\alpha_{2_o}} \int_{-\alpha_{2_o}}^{\alpha_{2_o}} d\alpha_2 \frac{1}{N} \int_{-\alpha_{1_o}}^{\alpha_{1_o}} d\alpha_1, \quad (2.22a)$$

$$\int [\mathcal{D}\phi(\tau)][\mathcal{D}p(\tau)] := \lim_{N \rightarrow \infty} \left( \prod_{n=1}^N \sqrt{\frac{2\sigma_n \sigma_{n-1}}{\sigma_n^2 + \sigma_{n-1}^2}} \right) \int \prod_{n=1}^{N-1} \frac{d\phi_n dp_{\phi_n}}{2\pi\hbar}, \quad (2.22b)$$

$$\int [\mathcal{D}v(\tau)][\mathcal{D}b(\tau)] := \lim_{N \rightarrow \infty} \left( \prod_{n=1}^N \sqrt{\frac{2d_n d_{n-1}}{d_n^2 + d_{n-1}^2}} \right) \int \prod_{n=1}^{N-1} \frac{dv_n db_n}{2\pi}. \quad (2.22c)$$

Ignoring the total derivatives with respect to  $\tau$  in Eqs. (2.20) and (2.21), we can read out the total effective Hamiltonian constraint as:

$$\begin{aligned} \mathcal{H}_{\text{eff}} = & -\frac{p_\phi^2}{\hbar^2} - \frac{1}{2\sigma^2} - 3\pi G \gamma^2 \left[ \left( v^2 + \frac{d^2}{2} \right) \left( \sin^2(2b) \left( 1 - \frac{4}{d^2} \right) + \frac{2}{d^2} \right) + iv \sin(4b) \left( 1 - \frac{4}{d^2} \right) \right] \\ & + \frac{3\pi G(1 + \gamma^2)}{4} \left[ \left( v^2 + \frac{d^2}{2} \right) \left( \sin^2(4b) \left( 1 - \frac{16}{d^2} \right) + \frac{8}{d^2} \right) + 2iv \sin(8b) \left( 1 - \frac{16}{d^2} \right) \right]. \end{aligned} \quad (2.23)$$

Note that  $\frac{d^2}{2}$  and  $\frac{2}{d^2}$  are the square of fluctuations of  $\hat{v}$  and  $\widehat{\sin(2b)}$  respectively. They can be seen as quantum corrections to the leading term:  $v^2 \sin^2(2b) + iv \sin(4b)$  of the *Euclidean* part  $\hat{\Theta}_E$  as well as the leading term:  $v^2 \sin^2(4b) + 2iv \sin(8b)$  of the *Lorentz* part.

A careful observation reveals that the real and imaginary parts of the leading terms can be synthesized into a *Moyal \*-product* [22], i.e.,

$$v^2 \sin^2(2b) + iv \sin(4b) = v e^{\frac{i}{2}(\overleftarrow{\partial}_v \overrightarrow{\partial}_b - \overleftarrow{\partial}_b \overrightarrow{\partial}_v)} (\sin(2b)v \sin(2b)) =: v * (\sin(2b)v \sin(2b)), \quad (2.24a)$$

$$v^2 \sin^2(4b) + 2iv \sin(8b) = v * (\sin(4b)v \sin(4b)). \quad (2.24b)$$

Therefore the effective Hamiltonian constraint takes the form:

$$\begin{aligned} \mathcal{H}_{\text{eff}} = & -\frac{p_\phi^2}{\hbar^2} - \frac{1}{2\sigma^2} - 3\pi G \gamma^2 \left[ v * (\sin(2b)v \sin(2b)) \left( 1 - \frac{4}{d^2} \right) + \frac{\sin^2(2b)d^2}{2} \left( 1 - \frac{4}{d^2} \right) + \frac{2v^2}{d^2} + 1 \right] \\ & + \frac{3\pi G(1 + \gamma^2)}{4} \left[ v * (\sin(4b)v \sin(4b)) \left( 1 - \frac{16}{d^2} \right) + \frac{\sin^2(4b)d^2}{2} \left( 1 - \frac{16}{d^2} \right) + \frac{8v^2}{d^2} + 4 \right]. \end{aligned} \quad (2.25)$$

The *Moyal \*-product* emerges in the gravitational part of the Hamiltonian, since both  $\hat{\Theta}_E \propto \hat{v}(\widehat{\sin(2b)})\hat{v}(\widehat{\sin(2b)})$  and  $\hat{\Theta}_L \propto \hat{v}(\widehat{\sin(4b)})\hat{v}(\widehat{\sin(4b)})$  are non-symmetric operators which can be regarded as a product of two self-adjoint

operators. Thus, in this model the coherent state functional integral also suggests the *Moyal \*-product* to express the effective Hamiltonian for the quantum system with a *non-symmetric* Hamiltonian operator. However, since the *Moyal \*-product* originates from the non-symmetry of the operator, one may doubt why we did not use a *symmetric* operator from the very beginning. To understand the motivation of the non-symmetric operator  $\hat{\Theta}$ , we recall that the initial Hamiltonian constraint operator in LQC is actually self-adjoint in the kinematical Hilbert space[8]. To resolve the constraint equation and find *physical* states, one feasible method is to rebuild the constraint equation as a *Klein-Gordon* like equation and treat the scalar  $\phi$  as an *internal time*. As a result, the constrained quantum system was recast into a relativistic particle whose dynamics is govern by a *Klein-Gordon* like equation with an *emergent time* variable [9]. The price to get this Klein-Gordon like equation is that the new gravitational Hamiltonian operator  $\hat{\Theta}$  becomes a multiplication of two self-adjoint operators, and hence it is no longer symmetric. But this does not indicate that one could not employ  $\hat{\Theta}$  in the intermediate step to find physical states. On the other hand, because the *Moyal \*-product* comes from the expectation value of the *multiplication* of two self-adjoint operators on coherent state, this non-symmetric  $\hat{\Theta}$  just provides a suitable arena to examine the *Moyal \*-product* from the path integral perspective.

We can also take another practical way to symmetrize  $\hat{\Theta}$  at the beginning. For example, one can define a symmetric version of  $\hat{\Theta}_E$  by

$$\hat{\Theta}'_E := \frac{1}{2}(\hat{\Theta}_E + \hat{\Theta}_E^\dagger) \propto [\widehat{\hat{v}(\sin(2b)\hat{v}\sin(2b))} + (\widehat{\sin(2b)\hat{v}\sin(2b)})\hat{v}], \quad (2.26)$$

and then carry out the same procedure of above coherent state functional integral. In the calculation of matrix element  $\langle \Psi_{\zeta_n} | \hat{\Theta}'_E | \Psi_{\zeta_{n-1}} \rangle$ , we could think that the operators  $\hat{v}$  and  $\widehat{\sin(2b)\hat{v}\sin(2b)}$  in  $\hat{\Theta}_E$  act on *bra*  $\langle \Psi_{\zeta_n} |$  and *ket*  $|\Psi_{\zeta_{n-1}}\rangle$  respectively, while  $\widehat{\sin(2b)\hat{v}\sin(2b)}$  and  $\hat{v}$  in  $\hat{\Theta}_E^\dagger$  act on *bra*  $\langle \Psi_{\zeta_n} |$  and *ket*  $|\Psi_{\zeta_{n-1}}\rangle$  respectively. Then it is not difficult to see that the imaginary parts generated by  $\hat{\Theta}_E$  and  $\hat{\Theta}_E^\dagger$  cancel each other. Hence for the symmetric Hamiltonian operator corresponding to  $\hat{\Theta}$ , we can finally get the following effective Hamiltonian constraint:

$$\mathcal{H} := -\frac{p_\phi^2}{\hbar^2} - \frac{1}{2\sigma^2} + 3\pi G \left( v^2 + \frac{1}{2\varepsilon^2} \right) \left( \sin^2(2b)(1 - (16 + 12\gamma^2)\varepsilon^2 - (1 + \gamma^2)(1 - 16\varepsilon^2)\sin^2(2b)) + 2\varepsilon^2 \right), \quad (2.27)$$

which takes the same form as (2.25) but without *\*-product* while  $\varepsilon \equiv 1/d$  denotes the quantum fluctuation of  $\sin b$ .

### III. EFFECTIVE DYNAMICS

Using the effective Hamiltonian constraint  $\mathcal{H}_{\text{eff}}$  which contains *Moyal \*-product*, one may investigate the corresponding dynamics by defining the evolution equation as:

$$\dot{f}(v, b) := \frac{1}{i\hbar} (f * \mathcal{H}_{\text{eff}} - \mathcal{H}_{\text{eff}} * f), \quad (3.1)$$

for any dynamical quantity  $f(v, b)$ . Especially, the evolution of basic variables can be obtained as:

$$\begin{aligned} \dot{v} = & -\frac{12\pi G\gamma^2}{\hbar} \left[ v * (v \sin(2b) \cos(2b)(1 - 4\varepsilon^2)) + \frac{\sin(2b) \cos(2b)(1 - 4\varepsilon^2)}{2\varepsilon^2} \right. \\ & \left. + \left( \frac{v^2}{2} - \left( v^2 + \frac{1}{2\varepsilon^2} \right) \sin^2(2b) - \frac{\sin^2(2b)(1 - 4\varepsilon^2)}{8\varepsilon^4} \right) \partial_b \varepsilon^2 \right] \\ & + \frac{12\pi G(1 + \gamma^2)}{4\hbar} \left[ 2v * (v \sin(4b) \cos(4b)(1 - 16\varepsilon^2)) + \frac{\sin(4b) \cos(4b)(1 - 16\varepsilon^2)}{\varepsilon^2} \right. \\ & \left. + \left( 2v^2 - \left( v^2 + \frac{1}{2\varepsilon^2} \right) 4\sin^2(4b) - \frac{\sin^2(4b)(1 - 16\varepsilon^2)}{8\varepsilon^4} \right) \partial_b \varepsilon^2 \right], \end{aligned} \quad (3.2a)$$

$$\begin{aligned} \dot{b} = & \frac{3\pi G}{\hbar} \left[ \left( 2v(1 - 4\varepsilon^2) \sin(2b) \right) * \sin(2b) + 4v\varepsilon^2 \right. \\ & \left. - \left( \frac{\sin^2(2b)}{2\varepsilon^4} (1 - 4\varepsilon^2) + \left( v^2 + \frac{1}{2\varepsilon^2} \right) 4\sin^2(2b) - 2v^2 \right) \partial_v \varepsilon^2 \right] \\ & - \frac{3\pi G(1 + \gamma^2)}{4\hbar} \left[ \left( 2v(1 - 16\varepsilon^2) \sin(4b) \right) * \sin(4b) + 16v\varepsilon^2 \right. \\ & \left. - \left( \frac{\sin^2(4b)}{2\varepsilon^4} (1 - 16\varepsilon^2) + \left( v^2 + \frac{1}{2\varepsilon^2} \right) 16\sin^2(2b) - 8v^2 \right) \partial_v \varepsilon^2 \right] \end{aligned} \quad (3.2b)$$

where  $\partial_b \varepsilon^2 \equiv \partial(\varepsilon^2)/\partial b$  and  $\partial_v \equiv \partial/\partial v$ . However, there seems no way to understand Eqs. (3.2a) and (3.2b) directly as effective classical equations because of the *\*-product* therein. Instead, we could use the effective Hamiltonian constraint (2.27) without *\*-product* to explore the effective dynamics. Using the conventional Poisson bracket, we can derive a modified Friedmann equation from the effective Hamiltonian (2.27) as:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_c}{3} \left[ \left(1 + \frac{1}{2v^2 \varepsilon^2}\right) \left( \sin(2b) \cos(2b) (1 - (16 + 12\gamma^2)\varepsilon^2 - 2(1 + \gamma^2)(1 - 16\varepsilon^2) \sin^2(2b)) \right. \right. \\ \left. \left. - \left( \sin^2(2b) (4(1 + \gamma^2) \cos^2(2b) - \gamma^2) - \frac{1}{2} \right) \partial_b \varepsilon^2 \right) \right. \\ \left. - \left( \sin^2 2b (1 - (16 + 12\gamma^2)\varepsilon^2 - (1 + \gamma^2)(1 - 16\varepsilon^2) \sin^2(2b)) + 2\varepsilon^2 \right) \frac{\partial_b \varepsilon^2}{8\varepsilon^2 v^2 \varepsilon^2} \right]^2 \quad (3.3)$$

where  $\rho_c \equiv \frac{\sqrt{3}}{32\pi^2 G^2 \hbar \gamma^3}$  is a constant. To annihilate  $\sin(2b)$  and  $\cos(2b)$  in Eq. (3.3), we use the constraint equation (2.27) to get

$$\sin^2(2b) = \frac{1 - (16 + 12\gamma^2)\varepsilon^2 - \sqrt{(1 - (16 + 12\gamma^2)\varepsilon^2)^2 - 4(1 + \gamma^2)(1 - 16\varepsilon^2)\chi}}{2(1 + \gamma^2)(1 - 16\varepsilon^2)^2}, \quad (3.4)$$

and

$$\chi \equiv \frac{K}{J} \frac{\rho}{\rho_c} - 2\varepsilon^2, \quad (3.5)$$

where  $\rho = \frac{p_\phi^2}{2V^2}$  is the density of matter,  $K \equiv 1 + \frac{\hbar^2}{2\sigma^2 p_\phi^2}$ , and  $J \equiv 1 + \frac{1}{2v^2 \varepsilon^2}$ . However, Eq. (3.3) looks problematic since it depends on the volume  $v$  of the chosen fiducial cell. This originates from the fact that we have to use the coherent states peaked on the phase points  $(v, b)$  in the path integral. In the final picture we have to remove the infrared regulator by letting the cell occupy full spatial manifold. In this limit, the irrelevant correction terms proportional to  $1/(v\varepsilon)^2$  could be neglected, while the relevant terms proportional to  $\varepsilon^2$  would be kept, since  $\varepsilon$  was understood as the fluctuation of  $\sin b$  which does not depend on the fiducial cell. We finally get

$$H^2 = \frac{8\pi G \rho_c}{3} \left[ \pm \frac{1}{2\beta} \sqrt{\left(\lambda - \sqrt{\lambda^2 - 4\beta\chi}\right) \left(2\beta - \lambda + \sqrt{\lambda^2 - 4\beta\chi}\right) \left(\lambda^2 - 4\beta\chi\right)} \right. \\ \left. + \left(\frac{1}{2} - \frac{\lambda - \sqrt{\lambda^2 - 4\beta\chi}}{2\beta} \left(\frac{4(1 + \gamma^2)(2\beta - \lambda + \sqrt{\lambda^2 - 4\beta\chi})}{2\beta} - \gamma^2\right)\right) \partial_b \varepsilon^2 \right]^2 \quad (3.6)$$

where the positive and negative signs correspond to the expanding and contracting universe respectively. Here we use notations:  $\lambda \equiv 1 - (16 + 12\gamma^2)\varepsilon^2$  and  $\beta \equiv (1 + \gamma^2)(1 - 16\varepsilon^2)$  for a concise writing. Note that Eq. (3.6) implies significant departure from classical GR. For simplicity, we first consider the case  $\partial_b \varepsilon^2 = 0$  and see whether the *bounce* or *re-collapse* determined by  $H = 0$  could occur. Then it is obvious that, for a contracting universe, the so-called quantum bounce of LQC will occur when

$$\chi = \frac{\lambda^2}{4\beta}, \quad (3.7)$$

which means that  $\rho$  increases to  $\rho_{\text{boun}} \approx \frac{\rho_c}{4(1 + \gamma^2)}$  if  $\varepsilon^2$  is neglected. On the other hand, for an expanding universe, a re-collapse would occur when  $\chi = 0$ , or equivalently  $\rho$  decreases to  $\rho_{\text{coll}} \approx 2\varepsilon^2 \rho_c$ , which coincide with the result in canonical theory [18], where  $\partial_b \varepsilon^2$  was assumed as higher order term and hence neglected. As pointed out in Refs.[17, 18], the inferred re-collapse is almost in all probability as viewed from the parameter space characterizing the quantum fluctuation  $\varepsilon$ . Intuitively, as the universe expands unboundedly, the matter density would become so tiny that its effect could be comparable to that of quantum fluctuations of the space-time geometry. Then the Hamiltonian constraint may force the universe to contract back. It should be noted that the effective equation and hence the inferred effect of re-collapse are only valid with the coherent states of Gaussian type. Whether there is a similar result for other semiclassical states is still an interesting open issue. For example, one may consider the affine coherent states in the *affine quantum gravity* approach developed by Klauder [26, 27].

In the case when  $\partial_b \varepsilon^2$  could not be neglected, the bounce would also be approached for a contracting universe. Because  $\partial_b \varepsilon^2$  would not be bigger than the order of  $\mathcal{O}(\varepsilon^2)$ ,  $\chi$  could be infinitely close to the result in Eq. (3.7) and lead to  $H = 0$ . However, for an expanding universe, both of the two terms in the bracket of the right hand side of

Eq. (3.6) are non-negative in large scale. As a result, the Hubble parameter would always keep non-zero unless  $\partial_b \varepsilon^2$  approaches 0 asymptotically. Therefore, the inferred re-collapse might occur only if  $\partial_b \varepsilon^2$  approaches 0 asymptotically.

If we neglect all the higher-order quantum corrections:  $\frac{1}{2\sigma^2}$ ,  $\varepsilon^2$  and  $\partial_b \varepsilon^2$ , Eq. (3.5) would be simplified to  $\chi = \frac{\rho}{\rho_c}$ , and hence a *first-order* modified Friedmann equation could be obtained from Eq. (3.6) as:

$$H^2 = \frac{8\pi G\rho}{3} \left[ 1 - \frac{\gamma^2 + 4(1 + \gamma^2)\rho/\rho_c}{1 + \gamma^2} + \frac{\gamma^2 \rho_c}{2(1 + \gamma^2)^2 \rho} \left( 1 - \frac{4(1 + \gamma^2)\rho}{\rho_c} \right) \left( 1 - \sqrt{1 - \frac{4(1 + \gamma^2)\rho}{\rho_c}} \right) \right]. \quad (3.8)$$

Note that this first-order modified Friedmann equation is different from that in Refs.[8, 17]. But it coincides with the modified Friedmann equation in Ref.[18]. Hence Eq. (3.8) still contains the particular information of alternative dynamics. It is easy to see that if the matter density increase to  $\rho = \frac{\rho_c}{4(1+\gamma^2)}$ , Hubble parameter would be zero and the *bounce* could occur for a contracting universe. On the other hand, in the classical regime of large scale, we have  $\chi \ll 1$  for  $\rho \ll \rho_c$  and hence Eq. (3.8) reduces to the standard classical Friedmann equation:  $H^2 = 8\pi G\rho/3$ .

#### IV. SUMMARY

Since there are quantization ambiguities in constructing the Hamiltonian constraint operator in LQC, it is crucial to check whether the key features of LQC, such as the quantum bounce and effective scenario, are robust against the ambiguities. Moreover, since LQC serves as a simple arena to test ideas and constructions induced in the full LQG, it is important to implement those treatments from the full theory to LQC as more as possible. Unlike the usual treatment in spatially flat and homogeneous models, the Lorentz term has to be quantized in a form quite different from the Euclidean one in full LQG. For the above purpose, this kind of quantization procedure which kept the distinction of the Lorentz and Euclidean terms was proposed as alternative dynamics for LQC [18]. It was shown in the resulted canonical effective theory that the classical big bang is again replaced by a quantum bounce and it is possible for the expanding universe to re-collapse due to the quantum gravity effect by certain assumption. Hence it is desirable to study such kind of predictions from different perspective. Meanwhile, it is also desirable to study the *Moyal \*-product* by coherent state functional integral approach within LQC models.

To carry out the above ideas, the present paper is devoted to study the coherent state functional integral in spatially flat isotropic FRW model coupled with a massless scalar field in the alternative dynamics framework of LQC. The main results can be summarized as follows. By the well-established canonical theory, the coherent state functional integral for LQC with alternative dynamics has been formulated by group averaging. For the non-symmetric gravitational Hamiltonian constraint operator, the *Moyal \*-product* emerges naturally in the resulted effective Hamiltonian with higher-order quantum corrections. For the corresponding symmetrized Hamiltonian operator, the effective Hamiltonian and modified Friedmann equation are also derived from the coherent state functional integral approach. It turns out that the quantum bounce resolution of big bang singularity can also be obtained by the path integral representation. On the other hand, if higher order corrections are included, there is a possibility for the re-collapse of an expanding universe due to the quantum gravity effect, which coincides with the result obtained in the canonical formalism. Moreover, the first-order modified Friedmann equation still contains the particular information of alternative dynamics and hence admits the possible phenomenological distinction between the different proposals of quantum dynamics. The alternative modified Friedmann equation (3.6) or (3.8) sets up a new arena for studying phenomenological issues of LQC.

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### Appendix A: Calculation of the functional integral

We give some details on the calculation of the Lorentz part of the matrix element of exponentiated gravitational Hamiltonian operator:  $\langle \Psi_{\zeta_n} | e^{-i\epsilon\alpha\hat{\Theta}_L} | \Psi_{\zeta_{n-1}} \rangle$ . The order of  $\mathcal{O}(\epsilon)$  of this matrix element is

$$\begin{aligned} \langle \Psi_{\zeta_n} | -i\epsilon\alpha\hat{\Theta}_L | \Psi_{\zeta_{n-1}} \rangle &= i\epsilon\alpha \frac{3\pi G(1+\gamma^2)}{16} \sum_k \left[ k(k+4)\Psi_{\zeta_n}^*(k)\Psi_{\zeta_{n-1}}(k+8) - 2k^2\Psi_{\zeta_n}^*(k)\Psi_{\zeta_{n-1}}(k) \right. \\ &\quad \left. + k(k-4)\Psi_{\zeta_n}^*(k)\Psi_{\zeta_{n-1}}(k-8) \right] \\ &\equiv i\epsilon\alpha \frac{3\pi G(1+\gamma^2)}{16} \left( L_{n,n-1}^+ - L_{n,n-1}^0 + L_{n,n-1}^- \right). \end{aligned} \quad (A1)$$

Now we need to deal with the three terms  $L_{n,n-1}^+, L_{n,n-1}^0, L_{n,n-1}^-$  separately. First, we get

$$\begin{aligned} L_{n,n-1}^+ &\equiv \sum_k (k^2 + 4k) e^{-\frac{(k-v_n)^2}{2d_n^2} - \frac{(k+8-v_{n-1})^2}{2d_{n-1}^2}} e^{-ib_n(k-v_n)+ib_{n-1}(k+8-v_{n-1})} \\ &= \exp\left(-\frac{8(v_n-v_{n-1})}{d_n^2+d_{n-1}^2} - \frac{32}{d_n^2+d_{n-1}^2} + i8\bar{b}_n - \frac{(v_n-v_{n-1})^2}{2(d_n^2+d_{n-1}^2)} + i\bar{b}_n(v_n-v_{n-1})\right) \\ &\quad \cdot \sum_k (k^2 + 4k) \exp\left(-(k-\bar{v}_n^+)^2/\bar{d}_n^2 - i(b_n-b_{n-1})(k-\bar{v}_n^+)\right), \end{aligned} \quad (A2)$$

where  $\bar{v}_n^+ \equiv \frac{d_{n-1}^2 v_n + d_n^2 (v_{n-1}-8)}{d_n^2 + d_{n-1}^2}$ . To do the summation in the above equation, we first have to rewrite  $k^2 + 4k$  as a function of  $k - \bar{v}_n^+$ :

$$\begin{aligned} k^2 + 4k &= (k - \bar{v}_n^+)^2 + (2\bar{v}_n^+ + 4)(k - \bar{v}_n^+) + \bar{v}_n^{+2} + 4\bar{v}_n^+ \\ &= (k - \bar{v}_n^+)^2 + 2\left(\bar{v}_n - \frac{6d_n^2 - 2d_{n-1}^2}{d_n^2 + d_{n-1}^2}\right)(k - \bar{v}_n^+) + \bar{v}_n^2 - \bar{v}_n \frac{12d_n^2 - 4d_{n-1}^2}{d_n^2 + d_{n-1}^2} + \frac{32d_n^2(d_n^2 - d_{n-1}^2)}{(d_n^2 + d_{n-1}^2)^2}, \end{aligned}$$

where  $\bar{v}_n \equiv \frac{d_{n-1}^2 v_n + d_n^2 v_{n-1}}{d_n^2 + d_{n-1}^2}$ . Do the summation term by term, we get

$$L_{n,n-1}^+ = \langle \Psi_{\zeta_n} | \Psi_{\zeta_{n-1}} \rangle e^{-\frac{32}{d_n^2 + d_{n-1}^2}} e^{i8\bar{b}_n} \left( (\bar{v}_n)^2 - \bar{v}_n \frac{8d_n^2}{d_n^2 + d_{n-1}^2} + \frac{\bar{d}_n^2}{2} + P_{n,n-1}^+ \right), \quad (A3)$$

where  $P_{n,n-1}^+$  denotes a polynomial of  $v_n - v_{n-1}$ ,  $b_n - b_{n-1}$  and  $d_n - d_{n-1}$  without the zeroth order term. Here we have expanded the factor  $\exp\left(-\frac{4(v_n-v_{n-1})}{d_n^2+d_{n-1}^2}\right)$  in Eq. (A2) as  $1 - \frac{4(v_n-v_{n-1})}{d_n^2+d_{n-1}^2} + \dots$ . Except for the leading term 1, all the other terms can be conflated with  $P_{n,n-1}^+$ . Under the continuous limit  $N \rightarrow \infty$ , this  $P_{n,n-1}^+$  does not contribute to the effective action of gravity. It is easy to calculate  $L_{n,n-1}^0$  and  $L_{n,n-1}^-$  as follows:

$$L_{n,n-1}^0 = 2\langle \Psi_{\zeta_n} | \Psi_{\zeta_{n-1}} \rangle \left( (\bar{v}_n)^2 + \frac{\bar{d}_n^2}{2} + P_{n,n-1}^0 \right), \quad (A4)$$

$$L_{n,n-1}^- = \langle \Psi_{\zeta_n} | \Psi_{\zeta_{n-1}} \rangle e^{-\frac{32}{d_n^2 + d_{n-1}^2}} e^{-i8\bar{b}_n} \left( (\bar{v}_n)^2 + \bar{v}_n \frac{8d_n^2}{d_n^2 + d_{n-1}^2} + \frac{\bar{d}_n^2}{2} + P_{n,n-1}^- \right). \quad (A5)$$

Taking the expansion  $e^{\left(-\frac{8}{d_n^2 + d_{n-1}^2}\right)} = 1 - \frac{8}{d_n^2 + d_{n-1}^2} + \mathcal{O}\left(\frac{1}{d^4}\right)$  and neglecting the higher order terms than  $\left(\frac{1}{d^2}\right)$ , we can get the combination

$$\begin{aligned} &L_{n,n-1}^+ - L_{n,n-1}^0 + L_{n,n-1}^- \\ &= -4\langle \Psi_{\zeta_n} | \Psi_{\zeta_{n-1}} \rangle \left[ \left( (\bar{v}_n)^2 + \frac{\bar{d}_n^2}{2} \right) \left( \sin^2(4\bar{b}_n) \left( 1 - \frac{32}{d_n^2 + d_{n-1}^2} \right) + \frac{16}{d_n^2 + d_{n-1}^2} \right) \right. \\ &\quad \left. + 2i \sin(8\bar{b}_n) \bar{v}_n \left( 1 - \frac{32}{d_n^2 + d_{n-1}^2} \right) \frac{2d_n^2}{d_n^2 + d_{n-1}^2} \right] + P_{n,n-1}^{\text{grav}}, \end{aligned}$$

